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c-Extensions of the Petersen Geometry for M_{22}

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In this paper we consider a group G acting flag-transitively on a geometry with diagram $\overset{1}{\circ}\overset{c}{—}\overset{2}{\circ}—\overset{3}{\circ}\overset{P^*}{—}\overset{4}{\circ}$ such that the residue of an element of type 1 is the Petersen geometry for the group M_{22}. Explicitly, G is generated by finite subgroups P_1, P_2, P_3, P_4 such that

(a) $G \neq G_i := \langle P_j | j \neq i\rangle$ for $1 \leq i \leq 4$.
(b) For $B := P_1 \cap P_2 \cap P_3 \cap P_4$ we have $B_G = 1$, where $B_G := \bigcap_{g\in G} B^g$.
(c) If $K_i = B_{G_i}$, $G_{ij} = \langle P_k | k \neq i, j\rangle$ and $K_{ij} = B_{G_{ij}}$, then $G_1/K_1 \cong M_{22}$ or $\mathrm{Aut}(M_{22})$ and the structures of the G_i/K_i, $i \neq 1$, and the G_{ij}/K_{ij} are as indicated by the diagram.

The set $\{P_1, P_2, P_3, P_4\}$ is called a *parabolic system* for the group.†

We are going to prove the following

THEOREM. Let G be a group with a parabolic system belonging to the diagram $\overset{1}{\circ}\overset{c}{—}\overset{2}{\circ}—\overset{3}{\circ}\overset{P^*}{—}\overset{4}{\circ}$ where $G_1/K_1 \cong M_{22}$ or $\mathrm{Aut}(M_{22})$. Then G is isomorphic to one of the following groups:

(a) $2^{10}M_{22}, 2^{11}M_{22}, 2^{10}\,\mathrm{Aut}(M_{22}), 2^{11}\,\mathrm{Aut}(M_{22})$,
(b) $U_6(2), 2\cdot U_6(2), U_6(2):2, 2\cdot U_6(2):2$,
(c) M_{24}.

REMARK. It follows from the diagram that

$$|P_1 : B_{P_1}| = |P_4 : B_{P_4}| = 2 \quad \text{and} \quad P_2/B_{P_2} \cong P_3/B_{P_3} \cong \Sigma_3,$$
$$G_2/K_2 \cong \mathbb{Z}_2 \times \Sigma_5, \qquad G_3/K_3 \cong \Sigma_4 \times \mathbb{Z}_2 \quad \text{and} \quad G_4/K_4 \cong 2^3L_3(2),$$
$$K_{14} = K_4.$$

This implies that K_1K_4/K_1 and K_1K_4/K_4 are 2-groups. Hence, by [7, (1.13)–(1.15)] K_1 and K_4 are 2-groups. Since B/K_1 is also a 2-group, B and all K_i are 2-groups.

The following lemma is due to A. Pasini.

LEMMA 1. *We have* $K_1 = 1$.

PROOF. Let $\Gamma = \Gamma(G; G_i)$ be the coset geometry belonging to our parabolic system. Let us call the type-1 elements of Γ 'points' and the type-2 elements of Γ 'lines'. Then each line consists of two points and the relation 'having the same points' is an equivalence relation on the set of lines on the point α_1, where we identify the coset G_1 with α_1. Clearly, this equivalence relation is preserved by G_1. So the action of G_1/K_1 on $\mathrm{res}(\alpha_1)$ implies that there are no two lines with the same points. In particular, $G_p \cap G_q \leq G_l$ for two points p, q on a line l.

†The reader unfamiliar with diagram geometries is referred to [1] and [5].

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Now let $g \in K_1$, p be a point collinear to α_1 and l the line on α_1 and p. Then g fixes l and α_1, hence $g \in G_p$. Let k be a line on p such that l and k are incident to a common element x of type 3 and q the other point on k. Then x is incident to α_1 and there is a line through α_1 and q in $\mathrm{res}(x)$. So g fixes q and $g \in G_p \cap G_q \leq G_k$. This means that g fixes all lines in $\mathrm{res}(p)$ which are coplanar with l. The action $G_p/K_p \cong M_{22}$, resp. $\mathrm{Aut}(M_{22})$, on $\mathrm{res}(p)$ shows that $K_1 \leq K_p$, hence $K_1 = K_p = 1$. $\square$

So from now on we may assume $G_1 \cong M_{22}$ or $\mathrm{Aut}(M_{22})$. We are going to prove Theorem 1 by determining generators and relations for G and then applying the Todd–Coxeter algorithm in Magma [2]. We start with the following generators for G_1:

$$x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}, y.$$

All these elements are involutions, $\langle x_1, \ldots, x_{10}\rangle \cong M_{22}$, y is the outer automorphism and the following relations are satisfied:

$$\begin{gathered}
[x_1,x_2]=1,\quad [x_1,x_3]=1,\quad x_1^{-1}x_5^{-1}x_1x_5x_2=1,\quad x_1^{-1}x_6^{-1}x_1x_6x_3=1,\\
[x_1,x_7]=1,\quad [x_1,x_8]=1,\quad [x_1,x_9]=1,\quad [x_2,x_3]=1,\quad [x_2,x_5]=1,\\
[x_2,x_6]=1,\quad x_2^{-1}x_7^{-1}x_2x_7x_1=1,\quad x_2^{-1}x_8^{-1}x_2x_8x_3=1,\quad [x_2,x_9]=1,\\
[x_3,x_5]=1,\quad [x_3,x_6]=1,\quad [x_3,x_7]=1,\quad [x_3,x_8]=1,\\
x_3^{-1}x_9^{-1}x_3x_9x_2=1,\quad [x_5,x_6]=1,\quad (x_5x_7)^3=1,\quad x_5^{-1}x_8^{-1}x_5x_8x_6=1,\\
[x_5,x_9]=1,\quad x_6^{-1}x_7^{-1}x_6x_7x_8=1,\quad [x_6,x_8]=1,\quad x_6^{-1}x_9^{-1}x_6x_9x_5=1,\\
[x_7,x_8]=1,\quad (x_7x_9)^4=1,\quad (x_8x_9)^3=1,\quad [x_{10},x_3]=1,\quad [x_{10},x_6]=1,\\
[x_{10},x_8]=1,\quad x_{10}^{-1}x_1^{-1}x_{10}x_1x_8=1,\quad x_{10}^{-1}x_2^{-1}x_{10}x_2x_6=1,\\
x_{10}^{-1}x_5^{-1}x_{10}x_5x_6x_3=1,\quad x_{10}^{-1}x_7^{-1}x_{10}x_7x_8x_3=1,\quad (x_9x_{10})^5=1,\quad [x_1,y]=1,\\
[x_2,y]=1,\quad [x_3,y]=1,\quad [x_5,y]=1,\quad [x_6,y]=1,\quad [x_7,y]=1,\\
[x_8,y]=1,\quad [x_9,y]=1,\quad y^{-1}x_{10}^{-1}yx_{10}x_3=1,\quad (x_1x_{10}x_7x_5x_9)^{11}=1.
\end{gathered}\tag{1}$$

Moreover, we have

$$\begin{aligned}
P_2 &= \langle x_1, x_2, x_3, x_5, x_6, x_7, x_8, y^i\rangle\\
P_3 &= \langle x_1, x_2, x_3, x_5, x_6, x_8, x_9, y^i\rangle\\
P_4 &= \langle x_1, x_2, x_3, x_5, x_6, x_8, x_{10}, y^i\rangle
\end{aligned}\tag{2}$$

and

$$B = \langle x_1, x_2, x_3, x_5, x_6, x_8, y^i\rangle \tag{3}$$

where $i = 1$ if $G_1 \cong \mathrm{Aut}(M_{22})$ and $i = 0$ otherwise. In the following, unless stated otherwise, we always assume $G_1 \cong \mathrm{Aut}(M_{22})$.

As remarked above, we have $G_2/K_2 \cong \mathbb{Z}_2 \times \Sigma_5$. Therefore, we can choose an element $x \in P_1$ such that $o(x) = 2$, $G = \langle G_1, x\rangle$ and $[x, G_{12}] \leq K_2$.

LEMMA 2. *We have* $[K_2, x] = 1$.

PROOF. Since $[x, G_{12}] \leq K_2 = C_{G_{12}}(K_2)$, G_{12} acts on $[K_2, x]$. But $|[K_2, x]| \leq 2^3$ and $K_2 \cap G_1'$, which is of order 2^4, is the only nontrivial G_{12}/K_2-submodule of K_2. So the assertion follows. $\square$

Let $Q_2 := \langle K_2, x\rangle$ and $H := \langle x_1, x_8, x_9, x_{10}\rangle \cong \Sigma_5$. By Lemma 2, Q_2 is elementary abelian of order 2^6 and $[H, Q_2] \le K_2$. With respect to the basis x_2, x_3, x_5, x_6, y, x of Q_2 the elements of H belong to 6×6 matrices over $GF(2)$ of the shape

$$\begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ * & \cdots & * & 1 \end{pmatrix},$$

where the 5×5 submatrix A is determined by the relations in (1). Since x is centralized by an element of order 3 in H, we may assume $[x, x_8x_9] = 1$. Using this and the above relations, it is not difficult to calculate that either $[x, H] = 1$ or

$$x_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 1 \end{pmatrix}, \qquad x_8 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$x_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad x_{10} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & d_2 & 0 & 1 & 0 & 1 \end{pmatrix},$$

with $a_5, d_2 \in \{0, 1\}$. In particular, $[x, x_8] = [x, x_9] = 1$ in any case.

Now look at $G_4 = \langle x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, y, x\rangle$. We already know that all generators of G_4 except x_7 and (possibly) x_1 commute with x. Moreover, we have $K_4 = \langle x_1, x_2, x_3, y\rangle$, $H_4 := \langle x_5, x_6, x_7, x_8, x_9\rangle \cong L_3(2)$ and $Q_4 = \langle K_4, x^{H_4}\rangle$, where $Q_4 = O_2(G_4)$. Using the relations in (1), we calculate that, with respect to the basis $\{x, x^{x_7}, x^{x_7x_9}\}$, we obtain the following matrices for the action of H_4 on Q_4/K_4:

$$x_5 = \begin{pmatrix} 1 & 0 & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{0} & \bar{1} \end{pmatrix}, \qquad x_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \bar{1} & \bar{0} & \bar{1} \end{pmatrix}, \qquad x_7 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \bar{1} & \bar{1} & \bar{1} \end{pmatrix},$$

$$x_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \bar{1} & \bar{1} & \bar{1} \end{pmatrix}, \qquad x_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Here we write $\bar{1}$, resp. $\bar{0}$, if we only know that the corresponding relations hold in the factor group Q_4/K_4 and we write 1, 0 if we know that they hold indeed.

Since there exists the involution x in $Q_4 \setminus K_4$ and an element of order 7 in $L_3(2)$ acts fixed-point freely on Q_4/K_4, it is easy to see that Q_4 is elementary abelian of order 2^7 if $[x, H] = 1$ or $a_5 = 0$, and that $Q_4 \cong D_8^3$ is extra special with $Z(Q_4) = \langle y\rangle$ if $a_5 = 1$.

Let us look at x_5. We have $(x^{x_7})^{x_5} = axx^{x_7}$ for some $a \in \langle x_1, x_2, x_3, y\rangle$. Now $x^{x_7x_5x_7} = x^{x_5x_7x_5} = x^{x_7x_5}$, hence $[a, x_7] \in \langle y\rangle$ and $a \in C_{K_4}(x_7) = \langle x_1, x_3, y\rangle$. Furthermore, $x^{x_7x_5x_6} = x^{x_6x_7x_8x_5} = x^{x_7x_5}$, so $a \in C_{K_4}(x_6) = \langle x_3, y\rangle$.

Suppose $a = by$ with $b \in \langle x_3\rangle$. Let $\tilde{x} = xy$ and substitute our basis by $\{\tilde{x}, \tilde{x}^{x_7}, \tilde{x}^{x_7x_9}\}$. Then we obtain the same matrices for x_5, x_6, x_7, x_8, x_9 and $\tilde{x}^{x_7x_5} = b\tilde{x}\tilde{x}^{x_7}$. This means that without loss of generality we may assume $a \in \langle x_3\rangle$.

It follows from the relations (1) that the action of x_5 on x^{x_7} already determines the action of H_4 on Q_4. Now we are going to use the Todd–Coxeter algorithm in Magma to calculate $|G : G_1|$ for the different cases. We have to distinguish whether $a = 1$ or $a = x_3$, and whether $[x, H] = 1$ or $[x, H] \neq 1$, where in the last case we have the open parameters a_5, d_2. We always apply Magma for both possibilities $G_1 \cong M_{22}$ and $G_1 \cong \mathrm{Aut}(M_{22})$ (except in the case $a_5 = 1$ which already implies $G_1 \cong \mathrm{Aut}(M_{22})$. We obtain the following results:

(1) If $[x, H] = 1$ and $a = 1$ then $|G : G_1| = 2^{11}$.
(2) If $[x, H] = 1$ and $a = x_3$ then $|G : G_1| = 2^9 \cdot 3^4$.
(3) If $[x, H] \neq 1$, $a_5 = 1$, $d_2 = 0$ and $a = 1$ then $|G : G_1| = 23 \cdot 11$.

In all other cases G is trivial.

In the first case $G \cong 2^{11} M_{22}$, resp. $2^{11}\,\mathrm{Aut}(M_{22})$, and $O_2(G)$ is the universal representation group of the P-geometry for M_{22}, see [4, Section 3]. We have $|Z(G)| = 2$ and $G/Z(G)$ as well has a parabolic system of the desired type. So we obtain case (a) of the theorem.

In the second case, we have $|G : G_1| = 2 \cdot |U_6(2) : M_{22}|$ and we have shown below (Example 1) that there exists a parabolic system for the group $U_6(2)$ satisfying these relations. This means that, if $G_1 \cong M_{22}$, then there exists a subgroup $N \trianglelefteq G$ such that $G/N \cong U_6(2)$. Now $|N| = 2$, and since $G = \langle G_1, G_4\rangle = \langle G_1', G_4'\rangle \leq G'$, we obtain a non-split extension $G \cong 2 \cdot U_6(2)$. If $G_1 \cong \mathrm{Aut}(M_{22})$ then $H := \langle G_1', G_4'\rangle \cong 2 \cdot U_6(2)$ and $|G : H| = 2$. Hence $G \cong 2 \cdot U_6(2) : 2$. Again in both cases $G/Z(G)$ also has such a parabolic system and we obtain (b).

In the third case we can use similar arguments to deduce from Example 2 that $G \cong M_{24}$, i.e., (c) $\square$

EXAMPLE 1. A parabolic system with diagram $\overset{1}{\circ}\overset{c}{\text{—}}\overset{2}{\circ}\text{—}\overset{3}{\circ}\overset{P^*}{\text{—}}\overset{4}{\circ}$ for the group $U_6(2)$.

We will give two different descriptions for the parabolic system: one by just giving matrices which satisfy the relations deduced above, and a second one which is more geometric and uses the action of $U_6(2) : 2 \leq Co_2$ on the Leech lattice.

Let V be a six-dimensional unitary space over $GF(4) = \{0, 1, \omega, \omega^2\}$, $G = SU_6(2)$ and choose a basis $\{v_1, v_2, v_3, w_3, w_2, w_1\}$ for V such that $\langle v_i, w_i\rangle$ are pairwise orthogonal hyperbolic planes. Then the following matrices belong to G:

$$x_1 = \begin{pmatrix} 1 & & & & & \\ & & & 1 & 0 & \\ & & & 0 & 1 & \\ & 1 & 0 & & & \\ & 0 & 1 & & & \\ & & & & & 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & & 1 & & & \\ 1 & & & 1 & & \\ 1 & & & & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$x_3 = \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 1 & & 1 & & & \\ 0 & & & 1 & & \\ 1 & & & & 1 & \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad x_5 = \begin{pmatrix} 1 & & & & & \\ \omega & 1 & & & & \\ \omega & & 1 & & & \\ \omega^2 & & & 1 & & \\ \omega^2 & & & & 1 & \\ 1 & \omega & \omega & \omega^2 & \omega^2 & 1 \end{pmatrix},$$

$$x_6 = \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ \omega & & 1 & & & \\ 0 & & & 1 & & \\ \omega^2 & & & & 1 & \\ 1 & \omega & 0 & \omega^2 & 0 & 1 \end{pmatrix}, \quad x_7 = \begin{pmatrix} 1 & 1 & 0 & 10 & 0 & \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$x_8 = \begin{pmatrix} 1 & & & & & \\ 0 & 1 & 0 & & & \\ 1 & 1 & 1 & & & \\ 0 & & & 1 & 0 & \\ 1 & & & 1 & 1 & \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad x_9 = \begin{pmatrix} 1 & & & & & \\ & 0 & 1 & & & \\ & 1 & 0 & & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ & & & & & 1 \end{pmatrix},$$

$$x_{10} = \begin{pmatrix} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ 1 & \omega & 1 & & & \\ & & & 1 & 0 & 0 \\ & & & \omega^2 & 1 & 0 \\ & & & 1 & 0 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ 1 & & & & & 1 \end{pmatrix}.$$

Moreover, one can calculate that their images in $\bar{G} := G/Z(G) \cong U_6(2)$ satisfy the relations in (1), resp. the relations we have deduced for x in the proof of Theorem 1. In particular, if we set

$$P_1 = \langle x_1, x_2, x_3, x_5, x_6, x_8, x\rangle, \qquad P_2 = \langle x_1, x_2, x_3, x_5, x_6, x_7, x_8\rangle,$$
$$P_3 = \langle x_1, x_2, x_3, x_5, x_6, x_8, x_9\rangle, \qquad P_4 = \langle x_1, x_2, x_3, x_5, x_6, x_8, x_{10}\rangle,$$

then $\{P_1, P_2, P_3, P_4\}$ is a parabolic system of the desired type. So it remains to show that $G = \langle P_1, P_2, P_3, P_4\rangle$.

Let $G_1 = \langle P_2, P_3, P_4\rangle$. Since $\bar{G}_1$ satisfies the relations in (1), we have $\bar{G}_1 \cong M_{22}$. On the other hand, by [3] M_{22} is a maximal subgroup of $U_6(2)$ and $x \notin G_1$. This implies the assertion.

Before starting with the second description we recall the definition of the P-geometry for Co_2 as it is given in [6] and some more facts about the action of this group on the Leech lattice.

Let Λ denote the Leech lattice with inner product (,) and let $\Lambda_2 = \{v \in \Lambda_2 | (v, v) = 32\}$. Consider the group $H := Co_2$ as the stabilizer in $\mathrm{Aut}(\Lambda)$ of a fixed vector $v_0 \in \Lambda_2$. Let $\Sigma = \{\{v, -v\} | v \in \Lambda_2, (v_0, v) = 0\}$. Then $H_\sigma \cong 2^{10} \cdot \mathrm{Aut}(M_{22})$ for $\sigma \in \Sigma$ and the orbits of H_σ on Σ are described in [6]. Define a graph on Σ in the way that σ and $\tau \in \Sigma$ are adjacent iff $H_\sigma \cap H_\tau \cong 2^9 \cdot 2^5 \cdot \Sigma_5$. A clique $\mathcal{C}$ in this graph is called closed iff for each $\{v, -v\}, \{w, -w\} \in \mathcal{C}$ there exists $\{u, -u\} \in \mathcal{C}$ such that $v_0 + v + w + u \in 2\Lambda$. It is shown in [6] that each closed clique is of size 1, 3, 7 or 15, and that, if we take as objects of type i the closed cliques of size $2^i - 1$ and define incidence by inclusion, than we obtain the P-geometry for Co_2.

Let $\Omega = \{\{v, v_0 - v\} | v \in \Lambda_2, (v, v_0) = 16\}$. Then H acts transitively on Ω and $H_\omega \cong U_6(2) : 2$ for $\omega \in \Omega$ see [6, Section 6]. By Table 2 of [6] H_σ has three orbits $\Omega_1, \Omega_2, \Omega_3$ on Ω and if $\sigma = \{u, -u\}$, $\omega = \{w, v_0 - w\}$ we may assume $(u, v) = 16, 0$, resp. 8. Moreover, for $\omega \in \Omega_3$ we have $H_\sigma \cap H_\omega \cong \mathrm{Aut}(M_{22})$. Since the orbits of H_σ on Ω bijectively correspond to the orbits of H_ω on Σ, for a fixed $\omega_0 = \{v, v_0 - v\} \in \Omega$, H_{ω_0} has exactly one orbit, say Σ_3, on Σ of length $|\Sigma_3| = |(U_6(2) : 2) : \mathrm{Aut}(M_{22})| = 20736$ which consists of the pairs $\{v, -v\} \in \Sigma$ such that $(v, w) = 8$. Now, for $i = 1, 2, 3, 4$, let

$$\Gamma_i = \{\mathcal{C} \cap \Sigma_3 | \mathcal{C} \text{ is a closed clique of size } 2^i - 1 \text{ in } \Sigma\}$$

and define an incidence relation between the Γ_i by inclusion. Considering the shapes of vectors from Λ_2 as given in [6] it is easy to see that $|x| = 2^{i-1}$ for $x \in \Gamma_i$ and that this really defines a geometry of the desired type with automorphism group $H_{\omega_0} \cong U_6(2) : 2$.

EXAMPLE 2. A geometry with diagram $\overset{1}{\circ}\overset{c}{—}\overset{2}{\circ}—\overset{3}{\circ}\overset{P^*}{—}\overset{4}{\circ}$ for the group M_{24}.

Let $\mathcal{S} = \mathcal{S}(24, 8, 5)$ be the Steiner system for $G = M_{24}$ with underlying set Ω. Let $\mathcal{O}$ denote the set of octads and $\mathcal{T}$ the set of trios. We assume that the reader is familiar with the action of G on $\mathcal{S}$, in particular we need the following property:

> Let $O \in \mathcal{O}$ and G_O the stabilizer of O in G. Then the complement $\Omega \setminus O$ bears the structure of an affine space and $G_O \cong 2^4 A_8$ acts as affine group $AGL(4, 2)$ on $\Omega \setminus O$.

Now we define a rank-4 geometry $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ by

$$
\begin{aligned}
\Gamma_1 &= \{\{p, q\} | p, q \in \Omega,\ p \neq q\} \\
\Gamma_2 &= \{\{a, b\} | a, b \in \Gamma_1,\ a \cap b = \emptyset\} \\
\Gamma_3 &= \{(O_1, P, \{O_2, O_3\}) | O_i \in \mathcal{O}, \{O_1, O_2, O_3\} \in \mathcal{T},\ P \text{ is a partition of } O_1 \\
&\qquad \text{into four parallel lines in the affine geometry on } \Omega \setminus O_2\} \\
\Gamma_4 &= \{(O, P) | O \in \mathcal{O},\ P \text{ is a partiton of } \Omega \setminus O \text{ into parallel lines}\}.
\end{aligned}
$$

We define incidences between the Γ_i as follows:

Let $x \in \Gamma_1$, $\{a, b\} \in \Gamma_2$ and $y \in \Gamma_3$ or Γ_4. Then x is incident to $\{a, b\}$ iff $x \in \{a, b\}$, x is incident to y iff $x \in P$ and $\{a, b\}$ is incident to y iff a and b are incident to y. Finally, $(O_1, P_1, \{O_2, O_3\}) \in \Gamma_3$ and $(O, P) \in \Gamma_4$ are incident iff $O = O_2$ or O_3 and $P_1 \subset P$.

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